Dual pairs and tensor categories of modules over Lie algebras $\widehat{gl}_\infty$ and $\mathcal{W}_{1+\infty}$

Weiqiang Wang

Department of Mathematics, Yale University, New Haven, CT 06520

Abstract

We introduce a tensor category $\mathcal{O}_+$ (resp. $\mathcal{O}_-$) of certain modules of $\widehat{gl}_\infty$ with non-negative (resp. non-positive) integral central charges with the usual tensor product. We also introduce a tensor category $(\mathcal{O}_f, \odot)$ consisting of certain modules over $GL(N)$ for all N. We show that the tensor categories $\mathcal{O}_\pm$ and $\mathcal{O}_f$ are semisimple abelian and all equivalent to each other. We give a formula to decompose a tensor product of two modules in each of these categories. We also introduce a tensor category $\mathcal{O}^w$ of certain modules over $\mathcal{W}_{1+\infty}$ with non-negative integral central charges. We show that $\mathcal{O}^w$ is semisimple abelian and give an explicit formula to decompose a tensor product of two modules in $\mathcal{O}^w$.

0   Introduction

A general important problem in representation theory is to study the decomposition of a tensor product of two representations of a Lie algebra $\mathfrak{g}$. For example, due to Weyl's complete reducibility theorem, a tensor product of two finite dimensional representations of a complex simple Lie algebra $\mathfrak{g}$ is decomposed into a direct sum of irreducibles.

However when one considers a similar problem in the representation theory of infinite dimensional Lie algebras, the story is usually much more complicated. For example, if one decomposes the usual tensor product of an irreducible integrable representation with a positive integral central charge with

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another irreducible integrable representation with another positive integral
central charge of an affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ [K], one gets an infinite di-
rect sum of irreducibles with infinite multiplicities. If one considers a similar
question for the negative central charge cases, one usually cannot expect to
get any reasonable decomposition due to the failure of complete reducibility.

Lie algebra $\widehat{gl}_\infty$, which throughout this paper will be denoted by $\widehat{gl}$ for
shortness, plays an important role in the representation theory of infinite
dimensional Lie algebras, with remarkable relations with solitons, KP equa-
tions, affine Kac-Moody algebras and the $\mathcal{W}_{1+\infty}$ algebra etc (cf. eg. [DJKM],
[KRa], [KR1]). In this paper, we introduce two tensor categories $\mathcal{O}_+$ and
$\mathcal{O}_-$ of representations of $\widehat{gl}$. The irreducible objects in category $\mathcal{O}_+$ (resp.
$\mathcal{O}_-$) are certain irreducible highest weight representations with non-negative
(resp. non-positive) integral central charges. Note that central charges in
each category $\mathcal{O}_\pm$ are not fixed.

We show that a tensor product of two irreducible modules in category
$\mathcal{O}_+$ (resp. $\mathcal{O}_-$) decomposes into an (infinite) direct sum of irreducibles with
FINITE multiplicities. We identify the irreducibles which appear in such a
decomposition and give the multiplicities of these irreducibles in terms of
certain multiplicities of irreducible representations appearing in the decom-
position of certain irreducible finite dimensional representations of a general
linear Lie group when restricted to some semisimple block-diagonal subgroup.
This is indeed a reciprocity law of branching rules due to the appearance of
the so-called "seesaw" pairs [Ku, H2].

We then show that the two categories $\mathcal{O}_\pm$ are semisimple abelian tensor
categories and are equivalent to each other. In the category $\mathcal{O}_+$ case, an im-
portant ingredient is a remarkable duality theorem first proved by I. Frenkel
[F1, FKRW] among representations of $\widehat{gl}$ with positive integral central charge
N and those of $GL(N)$. In the category $\mathcal{O}_-$ case, a similar duality theorem
from [KR2] is used. We also define $\mathcal{O}_f^N$ to be the category of all (possiblly
infinite dimensional) representations whose Jordan-Holder series consisting
of finite dimensional irreducible $GL(N)$-modules. Denote by $\mathcal{O}_f$ the category
which is a direct sum of categories $\mathcal{O}_f^N$ for all $N \geq 0$. We introduce a tensor
product $\odot$ on the category $\mathcal{O}_f$. and then establish an equivalence of tensor
categories between $\mathcal{O}_f$ and $\mathcal{O}_\pm$.

In [KR1], Kac and Radul systematically developed the theory of quasifi-
nite representations of $\widehat{\mathcal{D}}$ by using connections between $\widehat{\mathcal{D}}$ and $\widehat{gl}$ (or rather
some variations of $\widehat{gl}$) in an ingenious way. Here $\widehat{\mathcal{D}}$ is the universal central

extension of Lie algebra of differential operators on the circle, which is often referred to as $\mathcal{W}_{1+\infty}$ in literature. In this paper we define a semisimple abelian tensor category $\mathcal{O}^w$ of certain highest weight $\widehat{\mathcal{D}}$ modules with non-negative integral central charge. We prove that $\mathcal{O}^w$ is a semisimple abelian tensor category. Indeed this category consists of exactly all the representations of the vertex algebra $\mathcal{W}_{1+\infty}$ with non-negative integral central charges [FKRW]. Based on our tensor product decomposition theorem in category $\mathcal{O}_+$ and relations between $\widehat{gl}$ and $\widehat{\mathcal{D}}$, an explicit tensor product decomposition in the category $\mathcal{O}^w$ is presented. We remark that a similarly defined category of $\widehat{\mathcal{D}}$-modules with non-positive integral central charge will not be a semisimple tensor category.

Let us explain in more detail. In the case of positive (resp. negative) integral central charges, we will need N pairs of fermions $b^p(z), c^p(z)$ (resp. bosonic ghosts $\beta^p(z), \gamma^p(z)$), $p = 1, \ldots, N$. We will use bold letters when we want to treat both cases at the same time. For instance, we use $\mathbf{b}(z)$ (resp. $\mathbf{c}(z)$) to represent either $b(z)$ or $\beta(z)$ (resp. $c(z)$ or $\gamma(z)$). Denote by $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) the Fock space of a pair of fields $b(z), c(z)$ (resp. $\beta(z), \gamma(z)$). Now take N pairs of $\mathbf{bc}$ fields and consider the tensor product $\mathcal{F}_\pm^{\otimes N}$, namely the Fock space of N pairs of $\mathbf{bc}$ fields. Lie algebras $\widehat{gl}$ and $gl(N)$ acting on $\mathcal{F}_\pm^{\otimes N}$ naturally commute with each other and form a dual pair. The action of $gl(N)$ can be lifted to an action of $GL(N)$. Then with respect to the commuting actions of the dual pair $(GL(N), \widehat{gl})$, $\mathcal{F}_\pm^{\otimes N}$ decomposes into a direct sum of irreducible isotypic subspaces (cf. [F1], [FKRW], [KR2], see section 3 for notations):

$$\mathcal{F}_\pm^{\otimes N} = \bigoplus_{\lambda \in P_+^N} V_\lambda \bigotimes L(\Lambda_\pm(\lambda)),$$

where V_λ is the irreducible highest weight $GL(N)$-module of highest weight λ, and $L_{\pm N}(\Lambda_\pm(\lambda))$ is the irreducible $\widehat{gl}$-module with highest weight $\Lambda_\pm(\lambda)$ and central charge $\pm N$. We often write $(GL(N), \widehat{gl})$ as $\left(GL(N), \widehat{gl}|_{c=\pm N}\right)$ to emphasize that the action of $\widehat{gl}$ on $\mathcal{F}_\pm^{\otimes N}$ has central charge $\pm N$. Keeping the obvious isomorphism between $\mathcal{F}_\pm^{\otimes M} \otimes \mathcal{F}_\pm^{\otimes N}$ and $\mathcal{F}_\pm^{\otimes(M+N)}$ in mind, we see that the following two dual pairs act on the same Fock space $\mathcal{F}_\pm^{\otimes(M+N)}$:

$$\begin{cases} \Big(GL(M+N), & \widehat{gl}|_{c=\pm(M+N)}\Big) \\ \qquad \uparrow & \downarrow \\ \Big(GL(M) \times GL(N), & \widehat{gl}|_{c=\pm(M)} \bigoplus \widehat{gl}|_{c=\pm(N)}\Big) \end{cases}$$

where inclusions of Lie groups/algebras are shown by the arrows. Thus these two dual pairs form a seesaw pair [Ku, H2]. A consequence of a seesaw pair will be a duality among the branching rules corresponding to the two inclusions of Lie groups/algebras.

Note that the irreducible objects in categories $\mathcal{O}_{\pm}$ are precisely those irreducible $\widehat{gl}$-modules which appear in the decomposition of $\mathcal{F}_{\pm}^{\otimes N}(N \in \mathbb{N})$. By means of the seesaw pair above, we obtain an explicit formula for a decomposition of a tensor product of two irreducible modules in category $\mathcal{O}_{+}$ (resp. $\mathcal{O}_{-}$) into an infinite sum of irreducibles in $\mathcal{O}_{+}$ (resp. $\mathcal{O}_{-}$) with finite multiplicities. We further show that categories $\mathcal{O}_{\pm}$ are indeed semisimple abelian tensor categories and they are equivalent to each other. Combining with Frobenius reciprocity, we prove that the tensor category $(\mathcal{O}_f, \odot)$ is equivalent to the tensor categories $\mathcal{O}_{\pm}$.

Following [KR1], we have an injective homomorphism of Lie algebras $\widehat{\phi}_s$, $s \in \mathbb{C}$ from $\widehat{\mathcal{D}}$ to $\widehat{gl}$ such that the pull-back of an irreducible quasifinite $\widehat{gl}$-module via $\widehat{\phi}_s$ remains to be irreducible as a $\widehat{\mathcal{D}}$-module. We denote by $L_N(\widehat{\mathcal{D}}; \kappa(\lambda), s)$ the $\widehat{\mathcal{D}}$-module which is the pull-back via $\widehat{\phi}_s$ of the N-primitive module $L_N(\widehat{gl}, \Lambda_+(\lambda)), N \in \mathbb{N}$. A $\widehat{\mathcal{D}}$-module of the form of a finite tensor product $\bigotimes_i L_{n_i}(\widehat{\mathcal{D}}; \kappa(\lambda^i), s_i)$ is irreducible [KR1] if $\sum_i n_i = N, n_i \in \mathbb{N}$, $s_i \neq s_j \bmod \mathbb{Z}$, $i \neq j$, and $\lambda^i \in P_+^{n_i}$. We call an irreducible $\widehat{\mathcal{D}}$-module of such form primitive. Our category $\mathcal{O}^w$ of $\widehat{\mathcal{D}}$-module contains all the primitive $\widehat{\mathcal{D}}$-modules as irreducible objects. We present an explicit formula for a tensor product decomposition of modules in $\mathcal{O}^w$ and show that the category $\mathcal{O}^w$ is a semisimple abelian category.

The plan of this paper goes as follows. In Section 1 we review the Lie algebras $\widehat{gl}$ and its quasifinite hightest weight representations. We define the two categories $\mathcal{O}_{\pm}$ of $\widehat{gl}$-modules. In Section 2 we present duality theorems of the dual pair $(GL(N), \widehat{gl}|_{c=\pm N})$ with an integral central charge $\pm N$. In Section 3 we prove our theorem on the tensor product decomposition in the categories $\mathcal{O}_{\pm}$ and show that $\mathcal{O}_{\pm}$ and $\mathcal{O}_f$ are equivalent semisimple abelian categories. In section 4 we review the Lie algebra $\widehat{\mathcal{D}}$ and describe relations

between quasifinite representation theory of $\widehat{\mathcal{D}}$ and $\widehat{gl}$. We define a category $\mathcal{O}^w$ of $\widehat{\mathcal{D}}$-modules. In section 5 we prove our results on the tensor product decomposition in the category $\mathcal{O}^w$ and show that the category $\mathcal{O}^w$ is a semisimple abelian tensor category.

We expect that the dual pair principle [H1, H2] will have many applications in the representation theory of infinite dimensional Lie algebras. Our results can be generalized to other dual pair between a finite dimensional Lie group and an infinite dimensional Lie algebra. We will treat this in a future paper. We also hope our results will shed some lights on the constructions of Harish-Chandra modules of $\widehat{gl}$ and $\widehat{\mathcal{D}}$.

1   Categories $\mathcal{O}_{\pm}$ of $\widehat{gl}$-modules

Let us denote by gl the Lie algebra of all matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with only finitely many nonzero diagonals. Letting weight $E_{ij} = j - i$ defines a $\mathbb{Z}$–principal gradation $gl = \bigoplus_{j\in\mathbb{Z}} gl_j$. Denote by ν the automorphism of gl which sends E_{ij} to $E_{i+1,j+1}$ $(i, j \in \mathbb{Z})$. Denote by $\widehat{gl} = gl \bigoplus \mathbb{C}C$ the central extension given by the following 2–cocycle with values in $\mathbb{C}$:

$$C(A, B) = \operatorname{Tr}\left([J, A]B\right),$$

where $J = \sum_{j\leq 0} E_{ii}$. The $\mathbb{Z}$–gradation of Lie algebra gl extends to $\widehat{gl}$ by letting weight $C = 0$. In particular, we have the triangular decomposition

$$\widehat{gl} = \widehat{gl}_+ \oplus \widehat{gl}_0 \oplus \widehat{gl}_-,$$

where

$$\widehat{gl}_{\pm} = \bigoplus_{j\in\mathbb{N}} \widehat{gl}_{\pm j}, \quad \widehat{gl}_0 = gl_0 \oplus \mathbb{C}C.$$

Given $c \in \mathbb{C}$ and $\Lambda \in gl_0^*$, we let

$$\lambda_i = \Lambda(E_{ii}), \quad i \in \mathbb{Z}.$$

These λ_i's are called the labels of Λ. Let

$$H_i = E_{ii} - E_{i+1,i+1} + \delta_{i,0}C, \quad i \in \mathbb{Z},$$

and

$$h_i = \Lambda(H_i) = \lambda_i - \lambda_{i+1} + \delta_{i,0}c, \quad i \in \mathbb{Z}. \tag{1.1}$$

Denote by $L_c(\widehat{gl}, \Lambda)$ (or $L_c(\Lambda)$ if there is no ambiguity) the highest weight $\widehat{gl}$–module with highest weight Λ and central charge c. Easy to see that $L_c(\Lambda)$ is quasifinite (namely having finite dimensional graded subspaces according to the principal gradation of $\widehat{gl}$) if and only if all but finitely many $h_i, i \in \mathbb{Z}$ are zero.

Define an automorphism $\widehat{\nu}_k$ ($k \in \mathbb{N}$) by

$$
\begin{aligned}
\widehat{\nu}^k(E_{ij}) &= E_{i+k,j+k} \quad (i \neq j) \\
\widehat{\nu}^k(E_{ii}) &= E_{i+k,i+k} \quad (i > 0 \text{ or } i \leq -k) &\quad (1.2) \\
\widehat{\nu}^k(E_{ii}) &= E_{i+k,i+k} - C \quad (-k < i \leq 0) &\quad (1.3) \\
\widehat{\nu}^k(C) &= C.
\end{aligned}
$$

namely we have $\widehat{\nu}_k(E_{ij}) = E_{i+k,j+k}$ $(i \neq j), \widehat{\nu}_k(H_i) = H_{i+k}$. Clearly we have $\widehat{\nu}^k$ is the k-th composition of $\widehat{\nu} = \widehat{\nu}^1$. One can define the automorphism $\widehat{\nu}^k$ of $\widehat{gl}$ for $k \in -\mathbb{N}$ to be the inverse of $\widehat{\nu}^{-k}$.

Define $\Lambda_j \in gl_0^*, j \in \mathbb{Z}$ as follows:

$$
\Lambda_j(E_{ii}) = \begin{cases} 1, & \text{for} \quad 0 < i \leq j \\ -1, & \text{for} \quad j < i \leq 0 \\ 0, & \text{otherwise.} \end{cases} \quad (1.4)
$$

Define $\widehat{\Lambda}_0 \in \widehat{gl}_0^*$ by

$$
\widehat{\Lambda}_0(C) = 1, \quad \widehat{\Lambda}_0(E_{ii}) = 0 \text{ for all } i \in \mathbb{Z},
$$

and extend Λ_j from gl_0^* to $\widehat{gl}_0^*$ by letting $\Lambda_j(C) = 0$. Then

$$
\widehat{\Lambda}_j = \Lambda_j + \widehat{\Lambda}_0, \quad j \in \mathbb{Z}
$$

are the fundamental weights, i.e. $\widehat{\Lambda}_j(H_i) = \delta_{ij}$.

In the case that the central charge is a non-negative integer N, we call a highest weight Λ N-*primitive* and a $\widehat{gl}$-module $L(\Lambda)$ a N-*primitive module* if the h_i's (cf. equation(1.1)) satisfy

$$
h_k \in \mathbb{Z}_+ \ (k \in \mathbb{Z}), \quad \sum_k h_k = N. \quad (1.5)
$$

We denote by $\mathcal{O}_+$ the category of all $\widehat{gl}_0$-diagonalizable, $\widehat{gl}_+$-locally finite $\widehat{gl}$–modules, with N-primitive $\widehat{gl}$–modules for every $N \in \mathbb{Z}_+$ as all irreducible

objects and such that any module in $\mathcal{O}_+$ has a Jordan-Holder composition series in terms of N-primitive $\widehat{gl}$-modules ($N \in \mathbb{Z}_+$). Denote by $\mathcal{O}_+^N$ the subcategory of $\mathcal{O}_+$ consisting of those representations with central charge N. Here and further a $\mathfrak{g}$-module M is called $\mathfrak{g}'$-locally finite for a Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ if for any $a \in M$, the $\mathfrak{g}'$-submodule $\mathcal{U}(\mathfrak{g}')a$ is finite dimensional.

In the case that the central charge is a non-positive integer $-N$, we call a highest weight Λ to be $(-N)$-*primitive* and $L(\Lambda)$ a $(-N)$-*primitive module* if the h_i's satisfy

$$h_i \in \mathbb{Z}_+ \text{ if } i \neq 0 \text{ and } \sum_i h_i = -N. \tag{1.6}$$

$$\text{If } h_i \neq 0 \text{ and } h_j \neq 0, \text{ then } |i - j| \leq N. \tag{1.7}$$

Denote by $\mathcal{O}_-$ the category of all $\widehat{gl}_0$-diagonalizable, $\widehat{gl}_+$-locally finite $\widehat{gl}$ modules, with $(-N)$-primitive $\widehat{gl}$-modules for every $N \in \mathbb{Z}_+$ as all irreducible objects, and such that any module in $\mathcal{O}_-$ has a Jordan-Holder composition series in terms of $(-N)$-primitive $\widehat{gl}$-modules ($N \in \mathbb{Z}_+$).

Remark 1.1 *In the category $\mathcal{O}_+$, N-primitive $\widehat{gl}$-modules ($N \in \mathbb{Z}_+$) are exactly irreducible unitary highest weight $\widehat{gl}$-modules with respect to the standard compact anti-involution in $\widehat{gl}$.*

2 Duality of $(GL(N), \widehat{gl})$

We will first describe some Fock spaces on which the Lie algebras $gl(N)$ and $\widehat{gl}$ act on and commute with each other. For $\widehat{gl}$ with positive (resp. negative) integral central charge $c = N$ (resp. $c = -N$), we need N pairs of free fermions (resp. bosonic ghosts). We will deal with these two cases in a parallel way as follows. Let $\mathbf{b}(z)$ (resp. $\mathbf{c}(z)$) represent either $b(z)$ or $\beta(z)$ (resp. $c(z)$ or $\gamma(z)$). Introduce

$$\mathbf{b}(z) = \sum_{n \in \mathbb{Z}} \mathbf{b}(n)z^{-n}, \quad \mathbf{c}(z) = \sum_{n \in \mathbb{Z}} \mathbf{c}(n)z^{-n-1}.$$

We have the following commutation relations

$$[\mathbf{c}_m, \mathbf{b}_n]_\epsilon = \mathbf{c}_m \mathbf{b}_n + \epsilon \mathbf{b}_n \mathbf{c}_m = \delta_{m+n,0}$$

with $\epsilon = 1$ for fermions and $\epsilon = -1$ for bosons. We define the Fock space $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) of the fields $b(z)$ and $c(z)$ (resp. $\beta(z)$ and $\gamma(z)$), generated by the vacuum $|0\rangle$, satisfying

$$\mathbf{c}_n|0\rangle = \mathbf{b}_{n+1}|0\rangle = 0, \quad n \geq 0.$$

We denote by $\mathcal{F}_+^{\otimes N}$ (resp. $\mathcal{F}_-^{\otimes N}$) the Fock space of N pairs of fermions (resp. bosonic ghosts).

Now we take N pairs of **bc** fields, $\mathbf{b}^p(z), \mathbf{c}^p(z), p = 1, \ldots, N$, and consider the corresponding Fock space $\mathcal{F}_\pm^{\otimes N}$.

Introduce the following generating series

$$E(z, w) \equiv \sum_{i,j \in \mathbb{Z}} E_{ij} z^{i-1} w^{-j} = \epsilon \sum_{p=1}^{N} : \mathbf{c}^p(z)\mathbf{b}^p(w) : \qquad (2.8)$$

$$e^{pq}(z) \equiv \sum_{n \in \mathbb{Z}} e^{pq}(n) z^{-n-1} = \epsilon : \mathbf{c}^p(z)\mathbf{b}^q(z) :, \quad p, q = 1, \ldots, N, \qquad (2.9)$$

where the normal ordering $::$ means that the operators annihilating $|0\rangle$ are moved to the right and multiplied by $-\epsilon$.

It is well known (cf. e.g. [FKRW, KR2]) that the operators $E_{ij}, i, j \in \mathbb{Z}$ form a representation in $\mathcal{F}_\pm^{\otimes N}$ of the Lie algebra $\widehat{gl}$ with central charge ϵN; the operators $e^{pq}(n), p, q = 1, \ldots, N, n \in \mathbb{Z}$ form a representation of the affine Kac-Moody algebra $\widehat{gl(N)}$ with central charge ϵ [FF]. In particular the operators $e_{pq} := e^{pq}(0), p, q = 1, \cdots N$, form the horizonal subalgebra $gl(N)$ in $\widehat{gl(N)}$. One can also check directly that

$$[e_{pq}, E_{ij}] = 0, \quad p, q = 1, \ldots, N, \quad i, j \in \mathbb{Z}. \qquad (2.10)$$

The action of $gl(N)$ on $\mathcal{F}_\pm^{\otimes N}$ can be easily shown to lift to an action of $GL(N)$. Let $P_+^N = \{\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{Z}^N \mid \lambda_1 \geq \ldots \geq \lambda_N\}$. Denote by $\mathfrak{h}_N$ the diagonal matrices of the complex Lie algebra $gl(N)$. We denote by $^N V(\lambda)$ (or $V(\lambda)$ when there is no ambiguity) the irreducible representation of $GL(N)$ generated by a highest weight vector of highest weight λ which is fixed by the unipotent subgroup of upper triangular matrices with 1's along diagonals.

We define a map $\Lambda_+ : P_+^N \longrightarrow \widehat{gl}_0^*$:

$$\lambda = (\lambda_1, \cdots, \lambda_N) \longmapsto \Lambda_+(\lambda)$$

to be

$$\Lambda_+(\lambda) = \widehat{\Lambda}_{\lambda_1} + \cdots + \widehat{\Lambda}_{\lambda_N}. \tag{2.11}$$

We define another map $\Lambda_- : P_+^N \longrightarrow \widehat{gl}_0^*$:

$$\lambda = (\lambda_1, \cdots, \lambda_N) \longmapsto \Lambda_-(\lambda)$$

to be

$$\Lambda_-(\lambda) = (\ldots, 0, \lambda_{p+1}, \ldots, \lambda_N; \lambda_1, \ldots, \lambda_p, 0, \ldots), \tag{2.12}$$

where $p = 0$ if all $\lambda_i < 0$, $p = N$ if all $\lambda_i > 0$, and $1 \leq p < N$ is chosen such that $\lambda_p \geq 0 \geq \lambda_{p+1}$. Here the semicolon is put between the 0th and the first slots and gl_0^* is identified with $\mathbb{C}^\infty$ in a canonical way.

Then we have the following duality theorems.

Theorem 2.1 *With respect to the dual pair* $(GL(N), \widehat{gl}|_{c=N})$ *the module* $\mathcal{F}_+^{\otimes N}$ *decomposes into a direct sum of irreducible isotypic spaces:*

$$\mathcal{F}_+^{\otimes N} = \bigoplus_{\lambda \in P_+^N} V(\lambda) \otimes L_N\left(\Lambda_+(\lambda)\right) \tag{2.13}$$

where $V(\lambda)$ *is the irreducible highest weight* $GL(N)$*-module of highest weight* λ*, and* $L_N\left(\Lambda_+(\lambda)\right)$ *(See (2.11) for the definition of* $\Lambda_+(\lambda)$*) is the irreducible highest weight* $\widehat{gl}$*-module with central charge* N*.*

Theorem 2.1 was first proved in [F1] and also proved in [FKRW] in a different method. The following theorem is proved in [KR2].

Theorem 2.2 *With respect to the dual pair* $(GL(N), \widehat{gl}|_{c=N})$ *the module* $\mathcal{F}_-^{\otimes N}$ *decomposes as follows:*

$$\mathcal{F}_-^{\otimes N} = \bigoplus_{\lambda \in P_+^N} V(\lambda) \otimes L_{-N}\left(\Lambda_-(\lambda)\right) \tag{2.14}$$

where $V(\lambda)$ *is the irreducible highest weight* $GL(N)$*-module of highest weight* λ *and* $L_{-N}\left(\Lambda_-(\lambda)\right)$ *is the irreducible highest weight* $\widehat{gl}$*-module with central charge* $-N$ *(see (2.12) for the definition of* $\Lambda_-(\lambda)$*).*

3   Tensor product decomposition in $\mathcal{O}_\pm$

For a given $\lambda \in P_+^{M+N}(M, N \in \mathbb{N})$, when we restrict the irreducible representation $^{M+N}V(\lambda)$ of $GL(M+N)$ to the block-diagonal subgroup

$$\begin{bmatrix} GL(M) & 0 \\ 0 & GL(N) \end{bmatrix}$$

which is identified with $GL(M) \times GL(N)$, we have the following decomposition into a direct sum of irreducible $GL(M) \times GL(N)$-modules

$$^{M+N}V(\lambda) = \bigoplus_{\mu \in P_+^M, \nu \in P_+^N} C_{\mu\nu}^\lambda \, {}^M V(\mu) \bigotimes {}^N V(\nu) \tag{3.15}$$

where $C_{\mu\nu}^\lambda$ denotes the multiplicity in $^{M+N}V(\lambda)$ of the $GL(M) \times GL(N)$ module $^M V(\mu) \otimes {}^N V(\nu)$ ($\otimes$ here denotes the outer tensor product).

   Now we are ready to state our first main result.

Theorem 3.1 *The tensor product of two irreducible $\widehat{gl}$-modules in category $\mathcal{O}_+$ (resp. $\mathcal{O}_-$) can be decomposed into a direct sum of irreducible $\widehat{gl}$-modules with finite multiplicites. More precisely, given $\mu \in P_+^M, \nu \in P_+^N$, and consider the irreducible $\widehat{gl}$-modules $L_M(\Lambda_+(\mu))$ and $L_N(\Lambda_+(\nu))$, (resp. $L_{-M}(\Lambda_-(\mu))$ and $L_{-N}(\Lambda_-(\nu))$) with central charges M and N (resp. $-M$ and $-N$) in category $\mathcal{O}_+$ (resp. $\mathcal{O}_-$) (cf. equations (2.11), (2.12)). Then we have the following decomposition*

$$L_{\pm M}(\Lambda_\pm(\mu)) \bigotimes L_{\pm N}(\Lambda_\pm(\nu)) = \bigoplus_{\lambda \in P_+^{M+N}} C_{\mu,\nu}^\lambda L_{\pm(M+N)}(\Lambda_\pm(\lambda)) \tag{3.16}$$

where the multiplicity $C_{\mu\nu}^\lambda$ is defined as in (3.15), $L_{\pm(M+N)}(\Lambda_\pm(\lambda))$ is the irreducible $\widehat{gl}$-module in category $\mathcal{O}_\pm$ with highest weight $\Lambda_\pm(\lambda)$ and central charge $\pm(M+N)$.

Proof.   By the duality Theorems 2.1, 2.2, we have the following decompositions:

$$\mathcal{F}_\pm^{\otimes(M+N)} = \bigoplus_{\lambda \in P_+^{M+N}} {}^{M+N}V(\lambda) \bigotimes L_{\pm(M+N)}(\Lambda_\pm(\lambda)), \tag{3.17}$$

$$\mathcal{F}_\pm^{\otimes M} = \bigoplus_{\mu \in P_+^M} {}^M V(\mu) \bigotimes L_{\pm M}(\Lambda_\pm(\mu)), \tag{3.18}$$

$$\mathcal{F}_{\pm}^{\otimes N} = \bigoplus_{\nu \in P_+^N} {}^N V(\nu) \bigotimes L_{\pm N}\left(\Lambda_{\pm}(\nu)\right). \tag{3.19}$$

Obviously

$$\mathcal{F}_{\pm}^{\otimes(M+N)} = \mathcal{F}_{\pm}^{\otimes M} \bigotimes \mathcal{F}_{\pm}^{\otimes N}. \tag{3.20}$$

Due to (3.18) and (3.19), in addition to the dual pair

$$\left(GL(M+N), \widehat{gl}|_{c=M+N}\right), \tag{3.21}$$

we have another dual pair

$$\left(GL(M) \times GL(N), \widehat{gl}|_{c=M} \oplus \widehat{gl}|_{c=N}\right) \tag{3.22}$$

acting on the same Fock space $\mathcal{F}_{\pm}^{\otimes(M+N)}$. Clearly, we have the following inclusion relations of Lie algebras:

$$GL(M) \times GL(N) \subset GL(M+N), \quad \widehat{gl}|_{c=M+N} \subset_{diagonal} \widehat{gl}|_{c=M} \oplus \widehat{gl}|_{c=N}.$$

So the two dual pairs (3.21) and (3.22) form a seesaw pair in the sense of [Ku].

¿From (3.15) and (3.17), we have the following decomposition as modules over $GL(M) \times GL(N) \times \widehat{gl}|_{c=\pm(M+N)}$:

$$\mathcal{F}_{\pm}^{\otimes(M+N)} = \\ \bigoplus_{\lambda \in P_+^{M+N}} \bigoplus_{\mu \in P_+^M, \nu \in P_+^N} C_{\mu\nu}^{\lambda} \, {}^M V(\mu) \bigotimes {}^N V(\nu) \otimes L_{\pm(M+N)}\left(\Lambda_{\pm}(\lambda)\right). \tag{3.23}$$

¿From (3.18), (3.19) and (3.20), we have the following decomposition of the Fock space $\mathcal{F}_{\pm}^{\otimes(M+N)}$ as modules over $GL(M) \times GL(N) \times \widehat{gl}|_{c=\pm(M+N)}$:

$$\mathcal{F}_{\pm}^{\otimes(M+N)} = \\ \bigoplus_{\mu \in P_+^M, \nu \in P_+^N} {}^M V(\mu) \bigotimes {}^N V(\nu) \otimes L_{\pm M}\left(\Lambda_{\pm}(\mu)\right) \otimes L_{\pm N}\left(\Lambda_{\pm}(\nu)\right). \tag{3.24}$$

Now the theorem follows by comparing (3.23) with (3.24). $\square$

Remark 3.1 *The multiplicities $C_{\mu\nu}^{\lambda}$ can be computed by the combinatorial recipe known as Littlewood-Richardson Rule [M]. To some extend, one may view the dual pair $(GL(N), \widehat{gl})$ as some kind of $K \to \infty$ limit of dual pairs $(GL(N), gl(K))$ with a semi-infinite twist and Theorem 3.1 is a consequence of the stability for tensor product [H2], although the Lie algebra $\widehat{gl}$ is quite different from the direct limit of $gl(K)$ as $K \to \infty$.*

Note that a short exact sequence of $\widehat{gl}$-modules $0 \to V_1 \to V \to V_2 \to 0$, always splits if V_1, V_2 are two irreducible $\widehat{gl}$ modules with different cental charges. Since all irreducible modules from category $\mathcal{O}_+$ are integrable modules [K], the complete reducibility holds in $\mathcal{O}_+$. Now the complete reducibility in category $\mathcal{O}_-$ follows from Theorem 4.1 in [KR2]. Combining with Theorem 3.1, we have

Theorem 3.2 *The categories $\mathcal{O}_+$ and $\mathcal{O}_-$ are semisimple abelian tensor categories. Moreover, we have an equivalence of the two tensor categories $\mathcal{O}_+$ and $\mathcal{O}_-$ by letting the irreducible object $L_N(\Lambda_+(\lambda)) \in \mathcal{O}_+$ correspond to the irreducible object $L_N(\Lambda_-(\lambda)) \in \mathcal{O}_-$ for any $N \geq 0$ and $\lambda \in P_+^N$.*

Denote by $\mathcal{O}_f^N$ the category of $GL(N)$-modules which is a (possibly infinite) sum of finite dimensional irreducible modules. Denote by $\mathcal{O}_f$ the direct sum of the categories $\mathcal{O}_f^N$ for all $N \geq 0$, namely a category whose objects consist of a direct sum of modules of $GL(N)$ in $\mathcal{O}_f^N$ for all $N \geq 0$.

We introduce the following tensor product $\odot$ on the category $\mathcal{O}_f$: given a module $U \in \mathcal{O}_f^M$ and a module $V \in \mathcal{O}_f^N$, let $U \otimes V$ be the outer tensor product of U and V which is a $GL(M) \times GL(N)$-module. We define

$$U \bigodot V = \left(ind_{GL(M) \times GL(N)}^{GL(M+N)} U \bigotimes V \right)^{l.f.}$$

where $ind_H^G W$ denotes the induced G-module from a module W of $H \subset G$ and $X^{l.f.}$ denotes taking the locally finite vectors in X which are by definition the vectors which lie in some finite dimensional $GL(M+N)$-submodule of X.

Theorem 3.3 *We have an equivalence of tensor categories between $(\mathcal{O}_f, \odot)$ and $(\mathcal{O}_\pm, \otimes)$ by sending $^N V(\lambda)$ to $L_N \left(\widehat{gl}, \Lambda_\pm(\lambda) \right)$.*

Proof. It suffices to prove for $\mathcal{O}_+$ by Theorem 3.2. It is clear that the irreducible objects in the two categories $\mathcal{O}_f$ and $\mathcal{O}_\pm$ are in one-to-one correspondence.

Take a finite dimensional irreducible $GL(M+N)$-module $^{M+N} V(\lambda)$. By Frobenius Reciprocity we have

$$\text{Hom}_{GL(M+N)} \left(^{M+N} V(\lambda), \, ^M V(\mu) \bigodot \, ^N V(\nu) \right)$$
$$= \text{Hom}_{GL(M+N)} \left(^{M+N} V(\lambda), \, ind_{GL(M) \times GL(N)}^{GL(M+N)} \, ^M V(\mu) \bigotimes \, ^N V(\nu) \right)$$
$$= \text{Hom}_{GL(M) \times GL(N)} \left(^{M+N} V(\lambda) \mid_{GL(M) \times GL(N)}, \, ^M V(\mu) \bigotimes \, ^N V(\nu) \right).$$

So we have

$$\dim \operatorname{Hom}_{GL(M+N)} \left({}^{M+N}V(\lambda), {}^{M}V(\mu) \bigodot {}^{N}V(\nu) \right) = C^{\lambda}_{\mu\nu}.$$

Hence we have

$$ {}^{M}V(\mu) \bigodot {}^{N}V(\nu) = \bigoplus_{\lambda \in P^{M+N}_{+}} C^{\lambda}_{\mu,\nu} {}^{M+N}V(\lambda). $$

The tensor category $\mathcal{O}_f$ is clearly semisimple. Now the theorem follows by comparing with Theorem 3.1. □

Remark 3.2 *It follows that the category $\mathcal{O}_f$ is abelian as well. It is not difficult to show by using Frobenius reciprocity repeatedly that the tensor product of n ($n \geq 2$) irreducible modules in the category $\mathcal{O}_f$ (and also in $\mathcal{O}_{\pm}$ by Theorem 3.3) can be decomposed into an (infinite) direct sum of irreducibles with finite multiplicities. Indeed we can define a tensor subcategory of $\mathcal{O}_f$ (resp. $\mathcal{O}_{\pm}$) to be the smallest among subcategories of $\mathcal{O}_f$ (resp. $\mathcal{O}_{\pm}$) consisting of the same irreducible objects as in $\mathcal{O}_f$ (resp. $\mathcal{O}_{\pm}$) and closed under the finite direct sum and tensor product operations. Although an object in such a tensor subcategory may still be an infinite sum of irreducibles, the "infinity" behaves in a controlled way.*

Remark 3.3 *Consider the automorphism $\widehat{\nu}^{-k}$ on $\widehat{gl}$. Via pull-back the irreducible module $L(\sum_{a=1}^{N} \widehat{\Lambda}_{\lambda_a})$ in the category $\mathcal{O}^N_+$ becomes $L(\sum_{a=1}^{N} \widehat{\Lambda}_{\lambda_a+k})$. The corresponding operation on $\mathcal{O}^N_f$ by the equivalence of categories as in Theorem 3.3 is given by tensoring with $\det^{\otimes k}$, where $\det$ is the one-dimensional representation of $GL(N)$ of highest weight $(1, \ldots, 1)$.*

4 Category $\mathcal{O}^w$ of $\widehat{\mathcal{D}}$-modules

Let $\mathcal{D}$ be the Lie algebra of regular differential operators on the circle. The elements

$$ J^l_k = -t^{l+k}(\partial_t)^l, \quad l \in \mathbb{Z}_+, k \in \mathbb{Z} $$

form a basis of $\mathcal{D}$. $\mathcal{D}$ has also another basis

$$L_k^l = -t^k D^l, l \in \mathbb{Z}_+, k \in \mathbb{Z}$$

where $D = t\partial_t$. Denote by $\widehat{\mathcal{D}}$ the central extension of $\mathcal{D}$ by a one-dimensional center with a generator C, with commutation relations (cf. [KR1])

$$
\begin{aligned}
[t^r f(D), t^s g(D)] &= t^{r+s} \left(f(D+s)g(D) - f(D)g(D+r) \right) \\
&+ \Psi \left(t^r f(D), t^s g(D) \right) C
\end{aligned}
\tag{4.25}
$$

where

$$
\Psi \left(t^r f(D), t^s g(D) \right) = \begin{cases} \sum_{-r \leq j \leq -1} f(j)g(j+r), & r = -s \geq 0 \\ 0, & r + s \neq 0. \end{cases}
\tag{4.26}
$$

$\widehat{\mathcal{D}}$ is often refered to as the $\mathcal{W}_{1+\infty}$ algebra in literature.

Letting weight $J_k^l = k$ and weight $C = 0$ defines a principal gradation

$$\widehat{\mathcal{D}} = \bigoplus_{j \in \mathbb{Z}} \widehat{\mathcal{D}}_j \tag{4.27}$$

and so we have a triangular decomposition

$$\widehat{\mathcal{D}} = \widehat{\mathcal{D}}_+ \bigoplus \widehat{\mathcal{D}}_0 \bigoplus \widehat{\mathcal{D}}_- \tag{4.28}$$

where

$$\widehat{\mathcal{D}}_{\pm} = \bigoplus_{j \in \pm \mathbb{N}} \widehat{\mathcal{D}}_j, \quad \widehat{\mathcal{D}}_0 = \mathcal{D}_0 \bigoplus \mathbb{C} C.$$

Given $s \in \mathbb{C}$, $\mathcal{D}$ acts naturally on the vector space $t^s \mathbb{C}[t, t^{-1}]$. Choose a basis $v_j = t^{s+j}, j \in \mathbb{Z}$ of the space $t^s \mathbb{C}[t, t^{-1}]$. Then this gives rise to a monomorphism of Lie algebras

$$\phi_s : \mathcal{D} \longrightarrow gl, \ s \in \mathbb{C}$$

defined by

$$\phi_s \left(t^k f(D) \right) = \sum_{j \in \mathbb{Z}} f(s-j) E_{j-k,j}, \tag{4.29}$$

and then a monomorphism of Lie algebras [KR1]

$$\widehat{\phi}_s : \widehat{\mathcal{D}} \longrightarrow \widehat{gl}, \ s \in \mathbb{C} \qquad (4.30)$$

defined by

$$
\begin{aligned}
\widehat{\phi}_s \mid_{\widehat{\mathcal{D}}_j} &= \phi_s \mid_{\mathcal{D}_j} \quad \text{if} \quad j \neq 0, \\
\widehat{\phi}_s \left(e^{xD} \right) &= \phi_s \left(e^{xD} \right) - \frac{e^{sx} - 1}{e^x - 1} C, \\
\widehat{\phi}(C) &= C.
\end{aligned}
$$

Given a sequence of complex numbers $\xi = (\xi_j)_{j \in \mathbb{Z}_+}$ and a complex number c, there exists a unique irreducible highest weight $\widehat{\mathcal{D}}$-module $L_c(\widehat{\mathcal{D}}, \xi)$, which admits a nonzero vector v_ξ such that:

$$L_k^j v_\xi = 0 \quad \text{for} \quad k > 0, \quad L_0^j v_\xi = \xi_j v_\xi, \quad C = cI.$$

The module is called *quasifinite* if all eigenspaces of the operator D is finite dimensional. It was proved in [KR1] that $L_c(\widehat{\mathcal{D}}, \xi)$ is a quasifinite module if and only if the generating function

$$\Delta_\xi(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \xi_n$$

has the form

$$\Delta_\xi(x) = \frac{\phi(x)}{e^x - 1}$$

where

$$\phi(x) + c = \sum_i p_i(x) e^{r_i x} \quad \text{(a finite sum)}$$

for some nonzero polynomials in x such that $\sum_i p_i(0) = c$ and distinct complex numbers r_i. The numbers r_i are called the *exponents* of this module while the polynomial $p_i(x)$ are called their *multiplicities*. We call an irreducible quasifinite $\widehat{\mathcal{D}}$-module $L_c(\widehat{\mathcal{D}}, \xi)$ *primitive* if all multiplicities of its exponents are non-negative integers [FKRW]. Note that a primitive $\widehat{\mathcal{D}}$-module always has a non-negative integral central charge.

The following two propositions are taken from [KR1].

Proposition 4.1 *We have the isomorphism*

$$L_c(\widehat{\mathcal{D}}, \xi) \bigotimes L_{c'}(\widehat{\mathcal{D}}, \xi') \equiv L_{c+c'}(\mathcal{D}, \xi + \xi')$$

provided that no exponent of the first module is congruent mod $\mathbb{Z}$ to any exponent of the second module.

Proposition 4.2 *Let $L_c(\widehat{gl}, \Lambda)$ be the irreducible highest weight $\widehat{gl}$-module with highest weight $\Lambda = (\lambda_j)_{j \in \mathbb{Z}}$ and central charge c. $L_c(\widehat{gl}, \Lambda)$, when regarded as a $\widehat{\mathcal{D}}$-module via the embedding of Lie algebras $\widehat{\phi}_s$, is an irreducible quasifinite $\widehat{\mathcal{D}}$-module with exponents $s - j (j \in \mathbb{Z})$ of multiplicity h_j, where h_j is defined as in (1.1).*

Given $N > 0$ and $\lambda \in P_+^N$, $L_N(\Lambda_+(\lambda))$ is the highest weight $\widehat{gl}$-module with highest weight $\Lambda_+(\lambda)$ (*cf.* (2.11) for Λ_+) and central charge N. We will denote the $\widehat{\mathcal{D}}$-module $L_N\left(\widehat{gl}, \Lambda_+(\lambda)\right)$ in the sense of Proposition 4.2 by $L_N(\widehat{\mathcal{D}}; \kappa(\lambda), s)$ to emphasize its $\widehat{\mathcal{D}}$-module structure. All irreducible quasifinite $\widehat{\mathcal{D}}$-modules which we are concerned about in this paper are primitive and so have non-negative integral central charges.

Remark 4.1 *1). Any primitive $\widehat{\mathcal{D}}$-module with central charge N ($N \in \mathbb{Z}_+$) can be realized as the form of a finite tensor product*

$$\bigotimes_i L_{n_i}(\widehat{\mathcal{D}}; \kappa(\lambda^i), s_i),$$

where $\sum_i n_i = N, n_i \in \mathbb{N}$; $s_i \neq s_j \mod \mathbb{Z}, i \neq j$; $\lambda^i \in P_+^{n_i}$.
2). As $\widehat{\mathcal{D}}$-modules, $L_N\left(\widehat{\mathcal{D}}; \kappa(\lambda + l\mathbf{1}), s + l\right), l \in \mathbb{Z}$ are all isomorphic to each other, where $\mathbf{1} = (1, \cdots, 1) \in P_+^N$. These are all possible $\widehat{\mathcal{D}}$-module isomorphisms among $L_N(\widehat{\mathcal{D}}; \kappa(\lambda), s), \lambda \in P_+^N, s \in \mathbb{C}$ (cf. [KR1]).

We denote by $\mathcal{O}^w$ the category of $\widehat{\mathcal{D}}_0$-diagonalizable, $\widehat{\mathcal{D}}_+$-locally finite quasifinite $\widehat{\mathcal{D}}$-modules, with primitive modules as all irreducible objects and such that any module in $\mathcal{O}^w$ has a Jordan-Holder composition series in terms of primitive modules. Modules in category $\mathcal{O}^w$ always have non-negative integer central charges.

Remark 4.2 *The category $\mathcal{O}^w$ consists of exactly all representations of the vertex algebra $\mathcal{W}_{1+\infty}$ with non-negative integral central charges, cf. [FKRW].*

Remark 4.3 *If we try to study a similarly defined category of $\widehat{\mathcal{D}}$-modules with non-positive integral central charge, we will see the complete reducibility fails. Indeed such a category will contain irreducible quasifinite $\widehat{\mathcal{D}}$-modules $L_{-1}(\widehat{\mathcal{D}}; \kappa(0), s)$ $(s \in \mathbb{C})$ with generating functions $\Delta^s(x) = -\frac{e^{sx}-1}{e^x-1}$. But there exists non-split short exact sequences (cf. Proposition 4.2 in [W]):*

$$0 \longrightarrow L_{-1}(\widehat{\mathcal{D}}; \kappa(0), s) \longrightarrow M_{\pm} \longrightarrow L_{-1}(\widehat{\mathcal{D}}; \kappa(0), s \pm 1) \longrightarrow 0$$

for some $\widehat{\mathcal{D}}$-module $M_{\pm}$. Equivalently, there exists non-split short exact sequences of $\widehat{gl}$-modules with central charge -1:

$$0 \longrightarrow L_{-1}(\widehat{gl}, -\widehat{\Lambda}_i) \longrightarrow V_{\pm} \longrightarrow L_{-1}(\widehat{gl}, -\widehat{\Lambda}_{i\pm1}) \longrightarrow 0,$$

for some $\widehat{gl}$-module $V_{\pm}$, where $\widehat{\Lambda}_j$ means the j-th fundamental weight of $\widehat{gl}$.

5   Tensor product decomposition in category $\mathcal{O}^w$

Take two irreducible $\widehat{\mathcal{D}}$-modules V_1 and V_2 in the category $\mathcal{O}^w$. First let us consider the simplest case when all exponents in V_1 (resp. V_2) belong to the same congruence class mod $\mathbb{Z}$. Then by Remark 4.1, we may express V_1 and V_2 in the following forms:

$$V_1 = L_M(\widehat{\mathcal{D}}; \kappa(\mu), s), \quad V_2 = L_N(\widehat{\mathcal{D}}; \kappa(\nu), t).$$

Now we have two possibilities.

First, if the congruence class of exponents in $L_M(\widehat{\mathcal{D}}; \kappa(\mu), s)$ does not coincide with the one in $L_N(\widehat{\mathcal{D}}; \kappa(\nu), t)$, then the tensor product

$$L_M(\widehat{\mathcal{D}}; \kappa(\mu), s) \bigotimes L_N(\widehat{\mathcal{D}}; \kappa(\nu), t)$$

is an irreducible $\widehat{\mathcal{D}}$-module in category $\mathcal{O}^w$ by Proposition 4.1.

Secondly, if the congruence class of exponents in $L_N(\widehat{\mathcal{D}}; \kappa(\mu), s)$ coincides with the one in $L_N(\widehat{\mathcal{D}}; \kappa(\nu), t)$, we may assume $s = t$ by Remark 4.1. Then by Theorem 3.1 and Proposition 4.2, we have the following tensor product decomposition $V_1 \otimes V_2$ into a direct sum of irreducible $\widehat{\mathcal{D}}$-modules in category $\mathcal{O}^w$ (see (3.15) for notations):

$$L_M(\widehat{\mathcal{D}}; \kappa(\mu), s) \bigotimes L_N(\widehat{\mathcal{D}}; \kappa(\nu), s) = \bigoplus_{\lambda \in P_+^{M+N}} C_{\mu\nu}^{\lambda} L_{M+N}(\widehat{\mathcal{D}}; \kappa(\lambda), s) \quad (5.31)$$

In general, let $s_1, s_2, \cdots$ (resp. $t_1, t_2, \cdots$) be representatives of different mod $\mathbb{Z}$ congruence classes of the module V_1 (resp. V_2) with non-negative integral central charge M (resp. N). By Remark 4.1, we may write V_1 and V_2 in the following forms:

$$V_1 = \bigotimes_i L_{m_i}(\widehat{\mathcal{D}}; \kappa(\mu^i), s_i), \quad (5.32)$$

where $\sum_i m_i = M, m_i \in \mathbb{Z}_+; \mu^i \in P_+^{m_i}$; and

$$V_2 = \bigotimes_j L_{n_j}(\widehat{\mathcal{D}}; \kappa(\nu^j), t_j), \quad (5.33)$$

where $\sum_j n_j = N, n_j \in \mathbb{Z}_+; \lambda^j \in P_+^{n_j}$. We may assume (by rearrangement of the order of s_i's and t_j's if necessary) $s_a = t_a \bmod \mathbb{Z}$, $a = 1, \cdots, k$ for some $k \geq 0$. Here $k = 0$ would mean that none of the exponents $s_1, s_2, \cdots$ is congruent mod $\mathbb{Z}$ to any of the exponents $t_1, t_2, \cdots$. In this case, the tensor product $V_1 \otimes V_2$ is an irreducible $\widehat{\mathcal{D}}$-module in $\mathcal{O}_+$ by Proposition 4.1.

We may further assume $t_a = s_a, a = 1, \cdots, k$ by Remark 4.1. Then we have the following tensor product decomposition into a direct sum of irreducible $\widehat{\mathcal{D}}$-modules in $\mathcal{O}^w$ according to (5.31):

$$L_{m_a}(\widehat{\mathcal{D}}; \kappa(\mu^a), s_a) \bigotimes L_{n_a}(\widehat{\mathcal{D}}; \kappa(\nu^a), s_a)$$
$$= \bigoplus_{\lambda^a \in P_+^{m_a+n_a}} C_{\mu^a \nu^a}^{\lambda^a} L(\widehat{\mathcal{D}}; \kappa(\lambda^a), s_a), \quad a = 1, \cdots, k. \quad (5.34)$$

Now we are ready to state our main result of this section.

Theorem 5.1 *With notations as in* $(5.32, 5.33, 5.34)$, *we have the following decomposition of a tensor product of two modules V_1 and V_2 in $\mathcal{O}^w$ into the*

direct sum of irreducibles in $\mathcal{O}^w$:

$$
\begin{aligned}
V_1 \bigotimes V_2 \;=\; & \bigoplus_{\lambda^1 \in P_+^{m_1+n_1}} \cdots \bigoplus_{\lambda^k \in P_+^{m_k+n_k}} \{\Pi_{a=1}^k C_{\mu^a \nu^a}^{\lambda^a}\} \cdot \\
& \cdot \left\{ \bigotimes_{b=1}^{k} L(\widehat{\mathcal{D}}; \kappa(\lambda^b), s_b) \bigotimes \bigotimes_{i>k} L_{m_i}(\widehat{\mathcal{D}}; \kappa(\mu^i), s_i) \right. \\
& \left. \bigotimes \bigotimes_{j>k} L_{n_j}(\widehat{\mathcal{D}}; \kappa(\nu^j), t_j) \right\}.
\end{aligned}
\tag{5.35}
$$

Sketch of a proof. By (5.34), Propositions 4.1, 4.2 and Remark 4.1, we have

$$
\begin{aligned}
V_1 \bigotimes V_2 \;=\; & \bigotimes_{1 \le a \le k} \bigoplus_{\lambda^a \in P_+^{m_a+n_a}} \left\{ C_{\mu^a \nu^a}^{\lambda^a} L(\widehat{\mathcal{D}}; \kappa(\lambda^a), s_a) \right\} \bigotimes \\
& \bigotimes_{i>k} L_{m_i}(\widehat{\mathcal{D}}; \kappa(\mu^i), s_i) \bigotimes \bigotimes_{j>k} L_{n_j}(\widehat{\mathcal{D}}; \kappa(\nu^j), t_j).
\end{aligned}
\tag{5.36}
$$

Then it is easy to see that the right hand side of (5.36) is equal to the right hand side of (5.35). Each of the $\widehat{\mathcal{D}}$-modules appearing on the right hand side of (5.35) is irreducible by Proposition 4.1 and is in category $\mathcal{O}^w$ by Remark 4.1. □

Complete reducibility in Category $\mathcal{O}^w$ follows from complete reducibility in Category $\mathcal{O}^w$ with the help of Propositions 4.1, 4.2, and Remark 4.1. So we have proved the following theorem.

Theorem 5.2 $\mathcal{O}^w$ *is a semisimple abelian tensor category.*

Remark 5.1 *A tensor subcategory of $\mathcal{O}^w$ can be defined analogous to the tensor subcategory of $\mathcal{O}_\pm$ defined in Remark 3.2.*

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E-mail address: `wqwang@math.yale.edu`